So we compute and tabulate values of the difference quotient (the average rates of change) as follows:

| t	$\dfrac{P(t) - P(1992)}{t - 1992}$
1988	2,625,750
1990	2,781,000
1994	2,645,000
1996	2,544,250

☐ Another method is to plot the population function and estimate the slope of the tangent line when $t = 1992$. (See Example 5 in Section 2.7.)

From this table we see that $P'(1992)$ lies somewhere between 2,781,000 and 2,645,000. We estimate that the rate of increase of the population of the United States in 1992 was the average of these two numbers, namely

$$P'(1992) \approx 2.7 \text{ million people/year}$$

2.8 Exercises

1. On the given graph of f, mark lengths that represent $f(2)$, $f(2 + h)$, $f(2 + h) - f(2)$, and h. (Choose $h > 0$.) What line has slope $\dfrac{f(2 + h) - f(2)}{h}$?
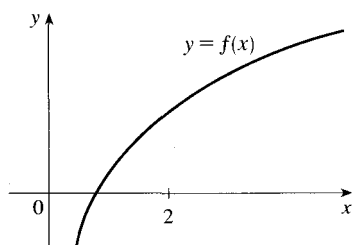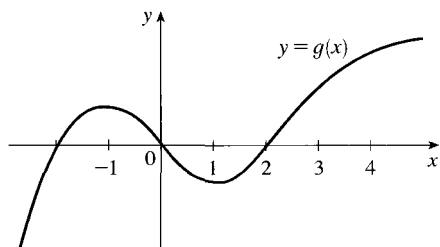

2. For the function f whose graph is shown in Exercise 1, arrange the following numbers in increasing order and explain your reasoning:

$$0 \qquad f'(2) \qquad f(3) - f(2) \qquad \tfrac{1}{2}[f(4) - f(2)]$$

3. For the function g whose graph is given, arrange the following numbers in increasing order and explain your reasoning:

$$0 \qquad g'(-2) \qquad g'(0) \qquad g'(2) \qquad g'(4)$$

4. If the tangent line to $y = f(x)$ at (4, 3) passes through the point (0, 2), find $f(4)$ and $f'(4)$.

5. Sketch the graph of a function f for which $f(0) = 0$, $f'(0) = 3$, $f'(1) = 0$, and $f'(2) = -1$.

6. Sketch the graph of a function g for which $g(0) = 0$, $g'(0) = 3$, $g'(1) = 0$, and $g'(2) = 1$.

7. If $f(x) = 3x^2 - 5x$, find $f'(2)$ and use it to find an equation of the tangent line to the parabola $y = 3x^2 - 5x$ at the point (2, 2).

8. If $g(x) = 1 - x^3$, find $g'(0)$ and use it to find an equation of the tangent line to the curve $y = 1 - x^3$ at the point (0, 1).

9. (a) If $F(x) = x^3 - 5x + 1$, find $F'(1)$ and use it to find an equation of the tangent line to the curve $y = x^3 - 5x + 1$ at the point (1, -3).
   (b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

10. (a) If $G(x) = x/(1 + 2x)$, find $G'(a)$ and use it to find an equation of the tangent line to the curve $y = x/(1 + 2x)$ at the point $\left(-\tfrac{1}{4}, -\tfrac{1}{2}\right)$.
    (b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

11. Let $f(x) = 3^x$. Estimate the value of $f'(1)$ in two ways:
    (a) By using Definition 2 and taking successively smaller values of h.
    (b) By zooming in on the graph of $y = 3^x$ and estimating the slope.

12. Let $g(x) = \tan x$. Estimate the value of $g'(\pi/4)$ in two ways:
    (a) By using Definition 2 and taking successively smaller values of h.

(b) By zooming in on the graph of $y = \tan x$ and estimating the slope.

13–18 ⌐ Find $f'(a)$.

13. $f(x) = 1 + x - 2x^2$

14. $f(x) = x^3 + 3x$

15. $f(x) = \dfrac{x}{2x - 1}$

16. $f(x) = \dfrac{x}{x^2 - 1}$

17. $f(x) = \dfrac{2}{\sqrt{3 - x}}$

18. $f(x) = \sqrt{3x + 1}$

. . . .

19–24 ⌐ Each limit represents the derivative of some function f at some number a. State f and a in each case.

19. $\displaystyle\lim_{h \to 0} \frac{\sqrt{1 + h} - 1}{h}$

20. $\displaystyle\lim_{h \to 0} \frac{(2 + h)^3 - 8}{h}$

21. $\displaystyle\lim_{x \to 1} \frac{x^9 - 1}{x - 1}$

22. $\displaystyle\lim_{x \to 3\pi} \frac{\cos x + 1}{x - 3\pi}$

23. $\displaystyle\lim_{t \to 0} \frac{\sin\left(\dfrac{\pi}{2} + t\right) - 1}{t}$

24. $\displaystyle\lim_{x \to 0} \frac{3^x - 1}{x}$

. . . .

25–26 ⌐ A particle moves along a straight line with equation of motion $s = f(t)$, where s is measured in meters and t in seconds. Find the velocity when $t = 2$.

25. $f(t) = t^2 - 6t - 5$

26. $f(t) = 2t^3 - t + 1$

. . . .

27. The cost of producing x ounces of gold from a new gold mine is $C = f(x)$ dollars.
(a) What is the meaning of the derivative $f'(x)$? What are its units?
(b) What does the statement $f'(800) = 17$ mean?
(c) Do you think the values of $f'(x)$ will increase or decrease in the short term? What about the long term? Explain.

28. The number of bacteria after t hours in a controlled laboratory experiment is $n = f(t)$.
(a) What is the meaning of the derivative $f'(5)$? What are its units?
(b) Suppose there is an unlimited amount of space and nutrients for the bacteria. Which do you think is larger, $f'(5)$ or $f'(10)$? If the supply of nutrients is limited, would that affect your conclusion? Explain.

29. The fuel consumption (measured in gallons per hour) of a car traveling at a speed of v miles per hour is $c = f(v)$.
(a) What is the meaning of the derivative $f'(v)$? What are its units?
(b) Write a sentence (in layman's terms) that explains the meaning of the equation $f'(20) = -0.05$.

30. The quantity (in pounds) of a gourmet ground coffee that is sold by a coffee company at a price of p dollars per pound is $Q = f(p)$.
(a) What is the meaning of the derivative $f'(8)$? What are its units?
(b) Is $f'(8)$ positive or negative? Explain.

31. Let $C(t)$ be the amount of U.S. cash per capita in circulation at time t. The table, supplied by the Treasury Department, gives values of $C(t)$ as of June 30 of the specified year. Interpret and estimate the value of $C'(1980)$.

t	1960	1970	1980	1990
$C(t)$	\$177	\$265	\$571	\$1063

32. Life expectancy has improved dramatically in the 20th century. The table gives values of $E(t)$, the life expectancy at birth (in years) of a male born in the year t in the United States. Interpret and estimate the values of $E'(1910)$ and $E'(1950)$.

t	$E(t)$	t	$E(t)$
1900	48.3	1950	65.6
1910	51.1	1960	66.6
1920	55.2	1970	67.1
1930	57.4	1980	70.0
1940	62.5	1990	71.8

33–34 ⌐ Determine whether or not $f'(0)$ exists.

33. $f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

34. $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

. . . .

Writing Project

Early Methods for Finding Tangents

The first person to formulate explicitly the ideas of limits and derivatives was Sir Isaac Newton in the 1660s. But Newton acknowledged that "If I have seen farther than other men, it is because I have stood on the shoulders of giants." Two of those giants were Pierre Fermat (1601–1665) and Newton's teacher at Cambridge, Isaac Barrow (1630–1677). Newton was familiar with the methods that these men used to find tangent lines, and their methods played a role in Newton's eventual formulation of calculus.